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Guide to Composition Functions

One of the most common struggles students deal with is understanding how to handle the composition of functions. However, this guide will cover the composition of two functions by examples and commentary. A part of analyzing composition functions is determining if any restrictions exist in the functions and what to do with them. This will be very handy when determining whether or not two functions are inverses of each other and determining the domain and ranges of functions and their compositions. But before anything can be done, it is key to understand the concept of substitution.

Substitution

By definition, substitution basically means the replacement of one thing for another. In mathematics, other keywords such as "replace" and "plug in" will work the same way as substitution. Essentially, composition functions are a matter of input and output via substitution of constants, variables, and functions. For example:

<u>What if we substituted a constant for *x* in this equation?</u>

$f(x)=x^2+x+1$

We have the equation assigned to $f(x)$. If we substitute the constant 2 for x, then it becomes the following:

$f(2)=(2)^2+(2)+1$
$f(2)=(4)+(2)+1$
$f(2)=7$

Notice the 2 appears in every place where x was. The 2 is the input and simplification of the equation is the output (7).

<u>As another example, let's substitute a variable for *x* this time.</u>

$f(x)=x^2+x+1$

We have the equation assigned to $f(x)$. If we substitute the variable q for x, then it becomes the following:

$f(q)=(q)^2+(q)+1$
$f(q)=q^2+q+1$

Notice the q appears in every place where x was. The q is the input and the simplification of the equation is the output.

<u>Now, what if we substituted a function that used the same variable as *f* (*x*)?</u>

$f(x)=x^2+x+1$

We have the equation assigned for $f(x)$. If we substitute a function such as "x–1" for x, then it becomes the following:

$f(x)=x^2+x+1$
$f(x-1)=(x-1)^2+(x-1)+1$
$f(x-1)=(x^2-2x+1)+(x-1)+1$
$f(x-1)=x^2-x+1$

Notice the x–1 appears in every place where x was. The x–1 is the input and the simplification of the equation is the output. ****Do not let the usage of the same variable as *x* fool you. Just make a complete substitution using the new function.****

Composition Of Two Functions

Now that we've covered substitution using a function, the substitution of one equation into another equation works similarly. The only differences are the notations. Suppose we have two different functions *f* (*x*) and *g* (*x*).

<u>Let's find the composition of *g* (*x*) into *f* (*x*).</u>

$f(x)=3x^2-2x+4$ and $g(x)=2x-3$

Let's say we wanted to substitute *g* (*x*) for *x* in *f* (*x*).

$$f(g(x))=3(g(x))^2-2(g(x))+4$$
$$f(2x-3)=3(2x-3)^2-2(2x-3)+4$$
$$f(2x-3)=3(4x^2-12x+9)-2(2x-3)+4$$
$$f(2x-3)=12x^2-36x+27-4x+6+4$$
$$f(2x-3)=12x^2-40x+37$$

Notice *g* (*x*) appears in every place where *x* was in *f* (*x*). Also notice that *f* (*x*) is the outer function and *g* (*x*) is plugged into it. What do you do then? Since *g* (*x*) equals 2*x*–3, you can substitute it to continue simplifying the equation.

$$f(g(x))=f\circ g(x)$$

Since *g* (*x*) was plugged into *f* (*x*), this indicates the composition of *g* (*x*) into *f* (*x*) which is read as *f* of *g* of *x* and is denoted as the following:

$$f\circ g(x)=12x^2-40x+37$$

Thus, the composition of *g* (*x*) into *f* (*x*) is the simplified equation with the appropriate notation.

<u>Let's now find the composition of *f* (*x*) into *g* (*x*).</u>

$f(x)=3x^2-2x+4$ and $g(x)=2x-3$

Let's say we wanted to substitute *f* (*x*) for *x* in *g* (*x*).

$$g(f(x))=2(f(x))-3$$
$$g(f(x))=2(3x^2-2x+4)-3$$
$$g(f(x))=(6x^2-4x+8)-3$$
$$g(f(x))=6x^2-4x+5$$

Notice *f* (*x*) appears in every place where *x* was in *g* (*x*). Also notice that *g* (*x*) is the outer function and *f* (*x*) is plugged into it. What do you do then? Substitute *f* (*x*) with its assigned function and continue to simplify.

$$g(f(x))=g\circ f(x)$$

Since *f* (*x*) was plugged into *g* (*x*), this indicates the composition of *f* (*x*) into *g* (*x*) which is read as *g* of *f* of *x* and is denoted as the following:

$$g\circ f(x)=6x^2-4x+5$$

Thus, the composition of *f* (*x*) into *g* (*x*) is the simplified equation with the appropriate notation.

The compositions of *g* into *f* and *f* into *g* have been shown in the previous examples. What about functions being compositions of themselves? This is possible and is performed by taking that function and replacing every *x* with that function.

Compositions Of A Function Into The Same Function

Suppose we are given two different functions *f* and *g*.

Let's find the composition of *f* (*x*) into *f* (*x*).

$f(x)=x-7$ and $g(x)=x^3+1$

Let's say we wanted to substitute *f* (*x*) for *x* in *f* (*x*). The composition of *f* into *f* is denoted like so:

$f \circ f(x)=f(f(x))$

This composition is equivalently written as *f* (*x*) into *f* (*x*). With that in mind, we can repeat the steps from the previous examples to perform this composition.

$f(f(x))=(f(x))-7$
$f(x-7)=(x-7)-7$
$f(x-7)=x-14$

Thus $f \circ f(x)=x-14$

Let's find the composition of *g* (*x*) into *g* (*x*).

$f(x)=x-7$ and $g(x)=x^3+1$

Let's say we wanted to substitute *g* (*x*) for *x* in *g* (*x*). The composition of *g* into *g* is denoted like so:

$g \circ g(x)=g(g(x))$

This composition is equivalently written as *g* (*x*) into *g* (*x*). Again, we can repeat the steps as in the composition of *f* into *f*.

$g(g(x))=(g(x))^3+1$
$g(x^3+1)=(x^3+1)^3+1$
$g(x^3+1)=(x^3+1)(x^3+1)(x^3+1)+1$ (Distributed / aka FOILed)
$g(x^3+1)=(x^3+1)(x^6+2x^3+1)+1$ (Distributed)
$g(x^3+1)=(x^9+3x^6+3x^3+1)+1$
$g(x^3+1)=x^9+3x^6+3x^3+2$

Thus $g \circ g(x)=x^9+3x^6+3x^3+2$

Restrictions

Restrictions are basically numerical values that can make an equation undefined (zero in the denominator) or imaginary (*i*). Restrictions occur commonly in equations when the variable is in the denominator and a numeric value could render it undefined. Imaginary numbers are considered restrictions because they do not exist on a real number plane. Imaginary numbers occur in equations when there is a negative number in an even-root function such as the square root of *x*, fourth root of *x*, sixth root of *x*, and so on.

Examples of equations with restrictions:

To find restrictions that would make an equation undefined, simply set the denominator NOT equal to zero.

$a(x)=\frac{32}{x}$

What value of x would make this equation undefined?

$x\neq 0$

Zero. This means that any real number, other than zero, to substitute in for x will give you an output of a real number.

$b(x)=\frac{3}{x-5}+2$

What value of x would make this equation undefined?

$x-5\neq 0$
$x\neq 5$

Positive five. This means that any real number,other than positive five, to substitute in for x will give you an output of a real number.

$c(x)=\frac{3x^2-5x+2}{x^2+4x-21}$

What values of x would make this equation undefined?

$x^2+4x-21\neq 0$
$(x+7)(x-3)\neq 0$
$x+7\neq 0$ and $x-3\neq 0$
$x\neq -7$ and $x\neq 3$

Set the denominator NOT equal to zero and then factor. Negative seven and positive three. This means that any real number, other than negative seven and positive three, to substitute in for x will output a real number.

To find restrictions of an equation concerning imaginary numbers, simply set the inside of the even-root greater than or equal to zero. Any number that is not included in the solution is a restriction.

$d(x)=\sqrt{x}$

What value of x would make this equation imaginary?

$x\geq 0$

Since the equation is the square root of x (an even-root function), it is ok to find the restriction. This means that any positive number and zero (nonnegative numbers) will work for this equation. So the restriction in this equation are any numbers less than zero (negative numbers) and not zero.

$e(x)=\sqrt[8]{x+24}$

What value of x would make this equation imaginary?

$x+24\geq 0$
$x\geq -24$

Since the equation is the eighth root of x (an even-root function), it is ok to find the restriction. This means that any number greater than or equal to negative twenty-four will work for this equation. So the restriction in this equation is any number less than negative twenty-four and not negative twenty-four.

Special Case

$f(x)=\sqrt{x^2-2x-24}$

What values of x would make this equation imaginary?

$x^2-2x-24=0$
$(x+4)(x-6)=0$
$x+4=0$ and $x-6=0$
$x=-4$ and $x=6$

Since the degree of the leading coefficient (2) inside the even-root is greater than 1, the inside function must be set equal to zero. Then test real numbers lesser and greater than the critical numbers to determine what values for x will output real numbers and what will output imaginary numbers. **The critical numbers are included in the solution.**

$x=-4$ and $x=6$

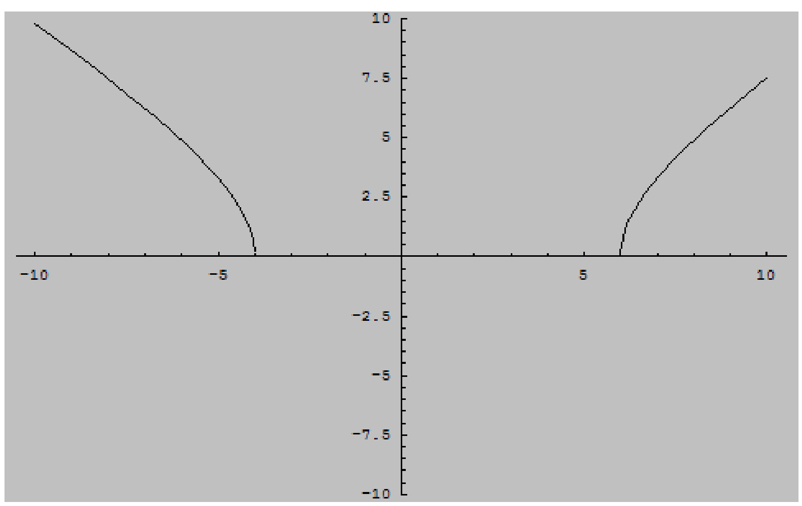
In this case, since negative four and positive six are the critical numbers for this equation, they are included in the solution. So a number less than negative four, a number greater than negative four and lesser than positive six, and a number greater than positive six must be tested in $f(x)$.

$x=-5$	-5 is less than -4	$f(-5)=\sqrt{11}$	Real
$x=-2$	-2 is greater than -4 and lesser than 6	$f(-2)=4i$	Imaginary
$x=7$	7 is greater than 6	$f(7)=\sqrt{11}$	Real

$x\leq-4$ and $x\geq6$

So any real number less than or equal to negative four and any real number greater than or equal to positive six are all numbers that will work i.e. output real numbers. <u>This means that our restriction is any number greater than negative four and less than positive six</u>.

Restriction:
$-4<x<6$



What about odd-roots? Will they give you restrictions of undefined or imaginary numbers? Generally, odd-roots are ok because the domain of any odd-root function is all real numbers (provided that the inside of the odd-root itself does not include a function that contains any restrictions).